

University of Groningen

## Model Reduction of a Flexible-Joint Robot: A Port-Hamiltonian Approach

Jardón Kojakhmetov, Hildeberto; Muñoz Arias, Mauricio; Scherpen, Jacquélien M.A.

*Published in:*  
IFAC-PapersOnLine, 2016

*DOI:*  
[10.1016/j.ifacol.2016.10.269](https://doi.org/10.1016/j.ifacol.2016.10.269)

**IMPORTANT NOTE:** You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2016

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Jardón Kojakhmetov, H., Muñoz Arias, M., & Scherpen, J. M. A. (2016). Model Reduction of a Flexible-Joint Robot: A Port-Hamiltonian Approach. In *IFAC-PapersOnLine, 2016* (pp. 832-837). (IFAC - Papers OnLine; Vol. 49, No. 18). IFAC. <https://doi.org/10.1016/j.ifacol.2016.10.269>

**Copyright**

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

**Take-down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

*Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.*

# Model reduction of a flexible-joint robot: a port-Hamiltonian approach

H. Jardón-Kojakhmetov\* M. Muñoz-Arias\*,\*\*  
 Jacqueliën M.A. Scherpen\*

\* *Engineering and Technology Institute (ENTEG), University of Groningen, Nijenborgh 4, 9747AG Groningen, The Netherlands.*  
 (e-mail: {h.jardon.kojakhmetov@rug.nl, m.munoz.arias, j.m.a.scherpen}@rug.nl)

\*\* *School of Electronics Engineering, Costa Rica Institute of Technology, P.O. Box 159-7050, Cartago, Costa Rica.*

**Abstract:** In this paper we explore the methodology of model order reduction based on singular perturbations for a flexible-joint robot within the port-Hamiltonian framework. We show that a flexible-joint robot has a port-Hamiltonian representation which is also a singularly perturbed ordinary differential equation. Moreover, the associated reduced slow subsystem corresponds to a port-Hamiltonian model of a rigid-joint robot. To exploit the usefulness of the reduced models, we provide a numerical example where an existing controller for a rigid robot is implemented.

© 2016, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.

## 1. INTRODUCTION

This document explores the model order reduction of a port-Hamiltonian (PH) system based on singular perturbations. The case study is a flexible-joint robot. By a series of transformations we show that a model for a flexible-joint robot can be written as 1) a PH system *and* 2) a singularly perturbed ordinary differential equation. Afterwards we show the effect of a composite control based on the reduced subsystems.

In the Euler-Lagrange (EL) framework, position control robotic systems have been thoroughly discussed in e.g., Canudas-de Wit et al. (1996); Murray et al. (1994); Ortega et al. (1998); Spong et al. (2006). In such a framework, the control design is based on selecting a suitable storage function that ensures position control. However, the desired storage function under the EL framework does not qualify as an energy function in any physical meaningful sense as stated in Canudas-de Wit et al. (1996); Ortega et al. (1998).

The PH modeling framework of van der Schaft and Maschke (2003); van der Schaft (2000) has received a considerable amount of interest in the last decade because of its insightful physical structure. It is well known that a large class of (nonlinear) physical systems can be described in the PH framework. The popularity of PH systems can be largely accredited to its application to analysis and control design of physical systems, e.g. Duindam et al. (2009); Fujimoto and Sugie (2001); van der Schaft and Maschke (2003); van der Schaft (2000). Control laws in the PH framework are derived with a clear physical interpretation via direct shaping of the closed-loop energy, interconnection, and dissipation structure, see Duindam et al. (2009); van der Schaft (2000).

On the other hand, model order reduction plays a crucial role in control design as well. Being able to synthesize controllers with a low number of variables is always more

convenient. Moreover, it is convenient as well to use reduction methods that preserve the structure of the original system, van der Schaft and Polyuga (2009); Polyuga and van der Schaft (2010); Scherpen and van der Schaft (2008). One of the several model reduction methods is based on singular perturbations, which is often applied to systems with two or more time-scales. It is known that under certain hyperbolicity properties (see Section 2) it is possible to obtain reduced models corresponding to systems in distinct time scales. The behavior of the full system can be inferred by an analysis of the reduced systems.

In the following sections we present a case study where model order reduction, based on singular perturbation, is applied to a PH system. In Section 2, we present a general background in the PH framework, especially for a class of standard mechanical systems. Furthermore, we recall the results of Viola et al. (2007) to equivalently describe the original PH system in a PH form which has a constant mass-inertia matrix in the Hamiltonian via a change of variables. This will prove helpful when writing the model of a flexible-joint robot. In Section 2 we also recall some basic properties of slow-fast systems. Afterwards in Section 3 we present a model of a flexible-joint robot that has a PH *and* a slow-fast structure. The corresponding reduced subsystem is the model of a rigid robot. We conclude this document with a simulation of a 2R flexible-joint robot for which a controller is designed based on the reduced models.

## 2. PRELIMINARIES

In this section we present the PH formalism for a class of standard mechanical systems. Additionally, we recapitulate the results of Fujimoto and Sugie (2001) in terms of generalized coordinates transformations for PH systems. We also recall the results of Viola et al. (2007) to transform the original system into PH form with a constant mass-

inertia matrix. Next, we include a brief introduction on slow-fast systems.

### 2.1 Port-Hamiltonian Systems

The PH framework is based on the description of systems in terms of energy variables, their interconnection structure, and power ports. PH systems include a large family of physical nonlinear systems. The transfer of energy between the physical system and the environment is given through energy elements, dissipation elements and power preserving ports, see Duindam et al. (2009); van der Schaft and Maschke (2003); van der Schaft (2000).

A class of PH system, introduced by van der Schaft and Maschke (2003), is described by

$$\Sigma = \begin{cases} \dot{x} = [J(x) - R(x)] \frac{\partial H(x)}{\partial x} + g(x) w \\ y = g(x)^\top \frac{\partial H(x)}{\partial x} \end{cases}$$

with states  $x \in \mathbb{R}^{\mathcal{N}}$ , skew-symmetric interconnection matrix  $J(x) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ , positive semi-definite damping matrix  $R(x) \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$ , and Hamiltonian  $H(x) \in \mathbb{R}$ . The matrix  $g(x) \in \mathbb{R}^{\mathcal{N} \times \mathcal{M}}$  weights the action of the control inputs  $w \in \mathbb{R}^{\mathcal{M}}$  on the system, and  $w, y \in \mathbb{R}^{\mathcal{M}}$  with  $\mathcal{M} \leq \mathcal{N}$ , form a power port pair.

In this preliminary, we restrict the analysis to the a class of standard mechanical systems with  $n$  ( $\mathcal{N} = 2n$ ) degrees of freedom (dof),

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & -D(q, p) \end{bmatrix} \begin{bmatrix} \frac{\partial H(q, p)}{\partial q} \\ \frac{\partial H(q, p)}{\partial p} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ G(q) \end{bmatrix} u \\ y &= G(q)^\top \frac{\partial H(q, p)}{\partial p} \end{aligned} \quad (1)$$

with generalized configuration coordinates  $q \in \mathbb{R}^n$ , generalized momenta  $p \in \mathbb{R}^n$ , damping matrix  $D(q, p) \in \mathbb{R}^{n \times n}$ , where  $D(q, p) = D(q, p)^\top \geq 0$ , output  $y \in \mathbb{R}^n$ , input  $u \in \mathbb{R}^n$ , and the input matrix  $G(q) \in \mathbb{R}^{n \times n}$ . The Hamiltonian function of (1) is given by

$$H(q, p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q) \quad (2)$$

where  $M(q) = M^\top(q) > 0$  is the  $n \times n$  inertia (generalized mass) matrix, and  $V(q)$  is the potential energy.

### 2.2 Nonconstant to constant mass-inertia matrix

Consider the class of standard mechanical systems in (1) with a nonconstant mass-inertia matrix  $M(q)$ . The aim of this section is to transform (1) into a PH system with a constant mass-inertia matrix by a generalized canonical transformation, see Fujimoto and Sugie (2001); Viola et al. (2007).

Consider the system in (1) with nonconstant  $M(q)$ , and a coordinate transformation  $(\bar{q}, \bar{p}) = \Phi(q, p)$  where

$$\Phi(q, p) = \begin{pmatrix} q - q^* \\ T(q)^{-1} p \end{pmatrix} = \begin{pmatrix} q - q_d \\ T(q)^\top \dot{q} \end{pmatrix} = \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} \quad (3)$$

with  $q^* \in \mathbb{R}^n$  being a constant position vector, and  $T(q)$  a lower triangular matrix such that

$$T(q) = T(\Phi^{-1}(\bar{q}, \bar{p})) = \bar{T}(\bar{q})$$

and

$$M(q) = T(q) T(q)^\top = \bar{T}(\bar{q}) \bar{T}(\bar{q})^\top$$

Consider now the Hamiltonian  $H(q, p)$  as in (2). Using (3), the new function  $\bar{H}(\bar{x}) = H(\Phi^{-1}(\bar{x}))$  and  $\bar{V}(\bar{q}) = V(\Phi^{-1}(\bar{q}))$  read as

$$\bar{H}(\bar{x}) = \bar{H}(\bar{q}, \bar{p}) = \frac{1}{2} \bar{p}^\top \bar{p} + \bar{V}(\bar{q})$$

Using this Hamiltonian and the coordinate transformation in (3), our system under consideration in (1) can be rewritten, as in van der Schaft and Maschke (2003), as follows

$$\begin{aligned} \begin{bmatrix} \dot{\bar{q}} \\ \dot{\bar{p}} \end{bmatrix} &= \begin{bmatrix} 0_{n \times n} & \bar{T}^{-\top} \\ -\bar{T}^{-1} & \bar{J}_2 - \bar{D} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}(\bar{q}, \bar{p})}{\partial \bar{q}} \\ \frac{\partial \bar{H}(\bar{q}, \bar{p})}{\partial \bar{p}} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ \bar{G} \end{bmatrix} \bar{v} \\ \bar{y} &= \bar{G}^\top \frac{\partial \bar{H}(\bar{q}, \bar{p})}{\partial \bar{p}} \end{aligned} \quad (4)$$

with a new input  $\bar{v} \in \mathbb{R}^n$ , and where the skew-symmetric matrix  $\bar{J}_2 = \bar{J}_2(\bar{q}, \bar{p})$  takes the form

$$\bar{J}_2(\bar{q}, \bar{p}) = \frac{\partial (\bar{T}^{-1} \bar{p})}{\partial \bar{q}} \bar{T}^{-\top} - \bar{T}^{-1} \frac{\partial (\bar{T}^{-1} \bar{p})^\top}{\partial \bar{q}} \quad (5)$$

with

$$(q, p) = \Phi^{-1}(\bar{q}, \bar{p})$$

together with the matrix  $\bar{D}(\bar{q}, \bar{p})$ , and the input matrix  $\bar{G}(\bar{q})$  given by

$$\begin{aligned} \bar{D}(\bar{q}, \bar{p}) &= \bar{T}(\bar{q})^{-1} D(\Phi^{-1}(\bar{q}, \bar{p})) \bar{T}(\bar{q})^{-\top} \\ \bar{G}(\bar{q}) &= \bar{T}(\bar{q})^{-1} G(\bar{q}) \end{aligned}$$

In (4) and (5) we have left out the argument  $\bar{q}$  of  $\bar{T}(\bar{q})$  for notational simplicity.

*Remark 1.* The change of coordinates described in this section is fundamental for the result of Section 3. Through this change of coordinates it is possible to write the model of a flexible joint robot as a PHS and a slow-fast system. Up to the authors' experience, if such a transformation is not performed, a slow-fast PHS is much more difficult to obtain.

### 2.3 Slow-fast systems

By a slow-fast system (SFS) we mean a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x, z, \varepsilon) \\ \varepsilon \dot{z} &= g(x, z, \varepsilon), \end{aligned} \quad (6)$$

where  $x \in \mathbb{R}^m$ ,  $z \in \mathbb{R}^n$  and  $\varepsilon > 0$  is a small parameter, i.e.,  $\varepsilon \ll 1$ , and where  $f$  and  $g$  are smooth functions. Note that due to the presence of the parameter  $\varepsilon$ , the variable  $z$  evolves much faster than  $x$ . Then,  $x$  and  $z$  are called the slow and the fast variables respectively. For  $\varepsilon \neq 0$  the new time parameter  $\tau = t/\varepsilon$  is defined. As a consequence, (6) is rewritten as

$$\begin{aligned} x' &= \varepsilon f(x, z, \varepsilon) \\ z' &= g(x, z, \varepsilon), \end{aligned} \quad (7)$$

where the prime denotes the derivative with respect to the re-scaled time  $\tau$ . Note that, for  $\varepsilon > 0$  and  $g$  not identically zero, the systems (6) and (7) are equivalent. A first approach to study the dynamics of slow-fast systems is to analyze the limit  $\varepsilon \rightarrow 0$  of (6) and (7). These limits

correspond to a Differential Algebraic Equation (DAE) given by

$$\begin{aligned}\dot{x} &= f(x, z, 0) \\ 0 &= g(x, z, 0),\end{aligned}\quad (8)$$

and to the layer equation

$$\begin{aligned}x' &= 0 \\ z' &= g(x, z, 0).\end{aligned}\quad (9)$$

Associated to those two limit equations, the critical manifold is defined as follows.

*Definition 1.* The critical manifold  $S$  is defined as

$$S = \{(x, z) \in \mathbb{R}^m \times \mathbb{R}^n \mid g(x, z, 0) = 0\}.$$

Note that the critical manifold  $S$  serves as the phase-space of the DAE (8) and as the set of equilibrium points of the layer equation (9).

If  $S$  is a set of hyperbolic points of (9), then  $S$  is called normally hyperbolic. Geometric Singular Perturbation Theory (GSPT), see e.g. Fenichel (1979); Kaper (1999); Jones (1995) shows that compact, normally hyperbolic invariant manifolds persist under  $C^1$ -small perturbations. In the present context, this means that if  $S_0 \subseteq S$  is a compact normally hyperbolic set, then, for  $\varepsilon > 0$  sufficiently small, there exists a normally hyperbolic invariant manifold  $S_\varepsilon$  of the slow-fast system (6) which is diffeomorphic to  $S_0$  and lies within distance of order  $O(\varepsilon)$  from  $S_0$ <sup>1</sup>. This implies that the flow along  $S_\varepsilon$  is approximately given by the flow of the DAE (8), along  $S_0$ .

Observe that if the matrix  $\partial_z g(x, z, 0)$  is regular on  $S$ , then by the implicit function theorem there exists a smooth function  $\phi$  such that  $S$  is given as a graph  $z = \phi(x)$ . Therefore, the flow along the critical manifold  $S$  is defined by

$$\dot{x} = f(x, \phi(x), 0), \quad (10)$$

which is called *the reduced slow vector field*.

Let  $x(t, x_0)$  be the flow defined by (10). The arguments of GSPT imply that the flow along the invariant manifold  $S_\varepsilon$  of (6) is given by  $x(t, x_0) + O(\varepsilon)$ . Moreover, assume that  $S$  is a set of *exponentially stable* equilibrium points of the layer equation (9). Then, there is a neighborhood  $D$  of  $S$  where all trajectories with initial condition in  $D$  are exponentially attracted to the invariant manifold  $S_\varepsilon$ .

In the context of control systems, the hyperbolicity property has been essential in the design of controllers based on model reduction. This is mainly because if the system does not have the hyperbolicity property, it cannot be decoupled into two (slow and fast) reduced subsystems.

Let us briefly recall the basic design methodology of composite control, for more details see Kokotovic et al. (1976); Kokotovic (1984); Kokotovic et al. (1986). Suppose we now study the control system

$$\begin{aligned}\dot{x} &= f(x, z, u, \varepsilon) \\ \varepsilon \dot{z} &= g(x, z, u, \varepsilon).\end{aligned}\quad (11)$$

The strategy is to consider the reduced systems

$$\dot{x} = f(x, \phi(x), u_s, 0), \quad (12)$$

and

$$z' = g(x, z, u_s + u_f, 0), \quad (13)$$

where  $u_s = u_s(x)$  denotes the controller for the reduced slow system (12) and  $u_f = u_f(x, z)$  the controller for the fast subsystem (13). The idea is to design controllers  $u_s$  and  $u_f$  that: 1) make the origin of the slow subsystem (12) exponentially stable, and 2) make the critical manifold  $z = \phi(x)$  exponentially stable. Then the controller  $u$  for the slow-fast system (11) is designed as  $u = u_s + u_f$ .

### 3. SLOW-FAST PORT-HAMILTONIAN MODEL

In this section, we derive a PH model of a flexible-joint robot which also has a slow-fast structure (6). In this way, the justification of designing controllers based on the reduced models is immediate. To start, let us make the following standard assumptions (Spong (1987); De Luca (2014)):

- All joints are of rotatory type
- The relative displacement (deflection) at each joint is small. Therefore we use a linear model for the springs.
- The  $i$ -th motor, which drives the  $i$ -th link, is mounted at the  $(i - 1)$ -th link.
- The center of mass of the motors are located along the rotation axes.
- The angular velocity of the motors is due only to their own spinning.

We denote by  $q_1 \in \mathbb{R}^n$  the links' angular positions and by  $q_2 \in \mathbb{R}^n$  the motors' angular displacement.

*Energies:* To obtain the Hamiltonian associated to the  $n$ -flexible-joint robot, let us first list the involved energies:

- Link's kinetic energy:  $K_l(q_1, \dot{q}_1) = \frac{1}{2} \dot{q}_1^T M_l(q_1) \dot{q}_1$ , where  $M_l(q_1) = M_l(q_1)^T > 0$ .
- Motor's kinetic energy:  $K_m(\dot{q}_2) = \frac{1}{2} \dot{q}_2^T I \dot{q}_2$ , where  $I = I^T > 0$  denotes the motors' inertia.
- Potential energy due to gravity:  $P_g(q_1) = \sum_{i=1}^n (P_{g,l_i}(q_1) + P_{g,m_i}(q_1))$ , where  $P_{g,l_i}$  and  $P_{g,m_i}$  are the potential energies due to the links and the motors, respectively.
- Potential energy due to joint stiffness:  $P_s(q_1, q_2) = \frac{1}{2} (q_1 - q_2)^T K (q_1 - q_2)$ , where  $K \in \mathbb{R}^{n \times n}$  is a symmetric, positive definite matrix of stiffness coefficients.

Now, let us assume that the stiffness coefficient is much higher than any other parameter of the system. This is we let  $K$  be defined as  $K = 1/\varepsilon \mathbb{I}_n$ , where  $\mathbb{I}_n$  denotes the  $n$ -dimensional identity matrix. Next, as it is customary, e.g. Spong (1987), let us define new coordinates as  $(q_1, \varepsilon z) = (q_1, q_1 - q_2)$ , and denote by  $q_\varepsilon \in \mathbb{R}^{2n}$  the generalized coordinates  $q_\varepsilon = (q_1, z)$ . Then the corresponding Hamiltonian  $H_\varepsilon = H_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon)$  can be written as

$$H_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) = \frac{1}{2} \dot{q}_\varepsilon^T M_\varepsilon(q_\varepsilon) \dot{q}_\varepsilon + V_\varepsilon(q_\varepsilon),$$

where  $M_\varepsilon(q_\varepsilon) \in \mathbb{R}^{2n \times 2n}$  and  $V_\varepsilon(q_\varepsilon) \in \mathbb{R}$  read as

$$M_\varepsilon(q_\varepsilon) = \begin{bmatrix} M_l(q_1) + I & -\varepsilon I \\ -\varepsilon I & \varepsilon^2 I \end{bmatrix}, \quad V_\varepsilon(q_\varepsilon) = P_g(q_1) + \frac{1}{2} \varepsilon z^T z$$

*Remark 2.* Note that  $H_0 = \lim_{\varepsilon \rightarrow 0} H_\varepsilon$  is precisely the Hamiltonian function of a rigid robot.

<sup>1</sup> Compactness ensures existence and uniqueness of  $S_\varepsilon$ , see Verhulst (2005).

Defining the generalized momenta as  $p_\varepsilon = M_\varepsilon(q_\varepsilon)\dot{q}_\varepsilon$ , it is straightforward to rewrite the Hamiltonian as  $H_\varepsilon(q_\varepsilon, p_\varepsilon) = \frac{1}{2}p_\varepsilon^T M_\varepsilon(q_\varepsilon)^{-1}p_\varepsilon + V_\varepsilon(q_\varepsilon)$ .

Let us now perform the canonical transformation as briefly exposed in Section 2.2. For this, a change of coordinates  $(\bar{q}_\varepsilon, \bar{p}_\varepsilon) = \Phi(q_\varepsilon, p_\varepsilon)$  is defined by

$$\begin{bmatrix} \bar{q}_\varepsilon \\ \bar{p}_\varepsilon \end{bmatrix} = \begin{bmatrix} q_\varepsilon \\ \bar{T}_\varepsilon(\bar{q}_\varepsilon)^T \dot{\bar{q}}_\varepsilon \end{bmatrix}, \quad (14)$$

where the matrix  $\bar{T}_\varepsilon(\bar{q}_\varepsilon) \in \mathbb{R}^{2n \times 2n}$  is lower triangular and is defined by  $M_\varepsilon = \bar{T}_\varepsilon(\bar{q}_\varepsilon)\bar{T}_\varepsilon(\bar{q}_\varepsilon)^T$ . Under this change of coordinates the Hamiltonian is rewritten as  $\bar{H}_\varepsilon = \frac{1}{2}\bar{p}_\varepsilon^T \bar{p}_\varepsilon + V(\bar{q}_\varepsilon)$ , and the port-Hamiltonian equations have the form (4), with  $(M, \bar{T}) = (M_\varepsilon, \bar{T}_\varepsilon)$ . In order to show that for a flexible-joint robot, the port-Hamiltonian equation takes the form of a slow-fast system we need to carefully study each term of (4). Let

$$\bar{T}_\varepsilon(\bar{q}_\varepsilon) = \begin{bmatrix} t_1 & 0 \\ t_2 & t_3 \end{bmatrix},$$

where  $\bar{T}_\varepsilon(\bar{q}_\varepsilon) \in \mathbb{R}^{2n \times 2n}$ , and for simplicity of notation we shall omit the dependence of  $t_i$  on  $\bar{q}_\varepsilon$ .

**Remark 3.** We assume that  $t_1$  is known because of the relation  $t_1 t_1^T = M_1(q_1) + I$ , which comes from a rigid robot model.

**Notation.** To save space, let us make the following definitions.

- First, let  $\bar{q}_\varepsilon = (\bar{q}_1, \bar{q}_2) = (q_1, z) \in \mathbb{R}^{2n}$  and  $\bar{p}_\varepsilon = (\bar{p}_1, \bar{p}_2) \in \mathbb{R}^{2n}$ .
- $T_1 = T_1(\bar{q}) \in \mathbb{R}^{n \times n}$  is defined by  $T_1 = t_1 t_1^T$ .
- $t_4 = t_4(\bar{q}) \in \mathbb{R}^{n \times n}$  is defined by  $t_4 t_4^T = I - I(t_1 t_1^T)^{-1}I = I - IT_1^{-1}I$ , which we assume to be known since  $t_1$  is known. It is a matter of simple linear algebra to show that  $t_4$  exists and is unique.
- $\alpha = \alpha(\bar{q}) \in \mathbb{R}^{n \times n}$  is defined by  $\alpha = t_4^{-1}I(t_1 t_1^T)^{-1}$ .
- $\beta = \beta(\bar{q}, \bar{p}) \in \mathbb{R}^{n \times n}$  is defined by  $\beta = \frac{\partial}{\partial \bar{q}_1}(t_1^{-1}\bar{p}_1)$ .
- $\gamma = \gamma(\bar{q}, \bar{p}) \in \mathbb{R}^{n \times n}$  is defined by  $\gamma = \frac{\partial}{\partial \bar{q}_1}(\alpha \bar{p}_1)$ .

It follows from careful computations that

$$\bar{T}_\varepsilon^{-1} = \begin{bmatrix} t_1^{-1} & 0 \\ \alpha & \frac{1}{\varepsilon} t_4^{-1} \end{bmatrix}, \quad \bar{J}_{2,\varepsilon}(\bar{q}_\varepsilon, \bar{p}_\varepsilon) = \begin{bmatrix} j_1 & j_2 \\ -j_2^T & j_3 \end{bmatrix}, \quad (15)$$

where

$$\begin{aligned} j_1 &= \beta t_1^{-T} - t_1^{-1} \beta^T, \\ j_2 &= \underbrace{-\beta \alpha^T - t_1^{-1} \gamma^T}_{=: j_{21}} - \frac{1}{\varepsilon} t_1^{-1} \left( \frac{\partial}{\partial \bar{q}_1}(t_4^{-1} \bar{p}_2) \right)^T = j_{21} - \frac{1}{\varepsilon} j_{22} \\ j_3 &= \underbrace{-\gamma \alpha^T + \alpha \gamma^T}_{=: j_{31}} - \\ &\quad \frac{1}{\varepsilon} \left( \frac{\partial}{\partial \bar{q}_1}(t_4^{-1} \bar{p}_2) \alpha^T - \alpha \left( \frac{\partial}{\partial \bar{q}_1}(t_4^{-1} \bar{p}_2) \right)^T \right)_{=: j_{32}} \\ &= j_{31} - \frac{1}{\varepsilon} j_{32}. \end{aligned} \quad (16)$$

**Remark 4.** Note that  $j_1 = -j_1^T$  and  $j_3 = -j_3^T$  and therefore  $\bar{J}_{2,\varepsilon}$  is indeed skew-symmetric.

**Proposition 1.** Under the coordinates  $(\bar{q}_\varepsilon, \bar{p}_\varepsilon)$  introduced above, a flexible-joint robot has the PH equations

$$\begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \\ \dot{\bar{p}}_1 \\ \dot{\bar{p}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_1^{-T} & \alpha^T \\ 0 & 0 & 0 & \frac{t_4^{-T}}{\varepsilon} \\ -t_1^{-1} & 0 & j_1 & j_{21} - \frac{j_{22}}{\varepsilon} \\ -\alpha^T & -\frac{t_4^{-1}}{\varepsilon} & -j_{21} + \frac{j_{22}}{\varepsilon} & j_{31} - \frac{j_{32}}{\varepsilon} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}_\varepsilon}{\partial \bar{q}_1} \\ \frac{\partial \bar{H}_\varepsilon}{\partial \bar{q}_2} \\ \frac{\partial \bar{H}_\varepsilon}{\partial \bar{p}_1} \\ \frac{\partial \bar{H}_\varepsilon}{\partial \bar{p}_2} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ 0_{n \times n} \\ \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} \bar{v}, \quad (17)$$

with  $G_i = G_i(\bar{q}_\varepsilon, \bar{p}_\varepsilon, \varepsilon) \in \mathbb{R}^{n \times n}$ , and where for notational simplicity we have left out the arguments of  $\bar{H}_\varepsilon, t_1, t_4, j_1, j_{21}, j_{22}, j_{31}$  and  $j_{32}$ . Moreover, the corresponding reduced slow subsystem has the dynamics of a rigid-joint robot in the PH framework.

**Proof.** The PH equations are obtained by substituting (15) and (16) into (4). Regarding the reduced systems, note that we can rewrite the PH equations as

$$\begin{aligned} \dot{\bar{q}}_1 &= t_1^{-T} \bar{p}_1 + \alpha^T \bar{p}_2 \\ \varepsilon \dot{\bar{q}}_2 &= t_4^{-T} \bar{p}_2 \\ \dot{\bar{p}}_1 &= -t_1^{-1} \frac{\partial \bar{H}_\varepsilon}{\partial \bar{q}_1} + j_1 \bar{p}_1 + \left( j_{21} - \frac{j_{22}}{\varepsilon} \right) \bar{p}_2 + \\ &\quad g_1(\bar{q}_\varepsilon, \bar{p}_\varepsilon) \bar{v} \\ \varepsilon \dot{\bar{p}}_2 &= -\varepsilon \alpha^T \frac{\partial \bar{H}}{\partial \bar{q}_1} - t_4^{-1} \frac{\partial \bar{H}}{\partial \bar{q}_2} - (\varepsilon j_{21} - j_{22}) \bar{p}_1 \\ &\quad + (\varepsilon j_{31} - j_{32}) \bar{p}_2 + \varepsilon g_2(\bar{q}_\varepsilon, \bar{p}_\varepsilon) \bar{v}. \end{aligned} \quad (18)$$

Taking now the limit  $\varepsilon \rightarrow 0$  of (18), we find the constraints

$$\begin{aligned} 0 &= \bar{p}_2 \\ 0 &= -t_4^{-1} \frac{\partial H_0}{\partial \bar{q}_2} + j_{22} \bar{p}_1 + j_{32} \bar{p}_2. \end{aligned}$$

Furthermore, note that  $\frac{\partial H_0}{\partial \bar{q}_2} = \varepsilon z$ ,  $j_{22} \in O(\varepsilon)$ ,  $\lim_{\varepsilon \rightarrow 0} q_\varepsilon = (q_1, 0)$ , and  $\lim_{\varepsilon \rightarrow 0} p_\varepsilon = ((M_1(q_1) + I)\dot{q}_1, 0)$ . Therefore the reduced system is simply the rigid PH model

$$\begin{bmatrix} \dot{q}_1 \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} 0 & t_1^{-T} \\ -t_1^{-1} & j_1 \end{bmatrix} \begin{bmatrix} \frac{\partial H_0}{\partial q_1} \\ \frac{\partial H_0}{\partial p_1} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ g_1(q_1, p_1) \end{bmatrix} u_s, \quad (19)$$

where  $u_s$  stands for the controller for the slow subsystem (12).

**Remark 5.** The PHS (19) is the PH model of a rigid robot, which physically means that the stiffness coefficient of the springs, of the flexible-joint robot, is infinity.

Since (17) has the structure of a slow-fast system ((with  $(\bar{q}_2, \bar{p}_2)$  being fast variables and  $(\bar{q}_1, \bar{p}_1)$  being the slow variables), we also find that the layer equation reads as

$$\begin{bmatrix} \bar{q}'_2 \\ \bar{p}'_2 \end{bmatrix} = \begin{bmatrix} 0 & t_4^{-T} \\ -t_4^{-1} & j_{32} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}}{\partial \bar{q}_2} \\ \frac{\partial \bar{H}}{\partial \bar{p}_2} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ g_2(\bar{q}_1, \bar{p}_1, \bar{q}_2, \bar{p}_2) \end{bmatrix} u_f, \quad (20)$$

where now  $(\bar{q}_1, \bar{p}_1)$  are fixed constants, and  $u_f$  stands for the controller of the layer equation (13). Note that the layer equation has also a PH structure.

With the previous exposition we have found reduced systems for a flexible-joint robot whose model is written in the PH framework. For control purposes, and following e.g., Kokotovic et al. (1986), it is possible to design controllers for a flexible-link robot from those for the rigid-robot and the layer equation.

#### 4. NUMERICAL EXAMPLE

To illustrate our previous exposition, we present in this section a simulation of a composite control of a 2R planar flexible-joint manipulator. Joint flexibility can be attributed to several physical factors, like motor-to-link coupling, harmonic drives, etc Spong (1987); De Luca (2014). For simplicity, we assume that the robot acts on the horizontal plane and thus, gravitational effects are neglected. A schematic of the 2R flexible-joint robot is shown in Figure 1.

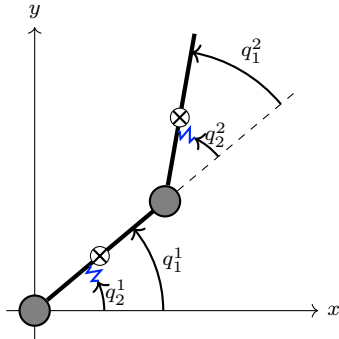


Fig. 1. Schematic of a 2R planar, flexible-joint robot.

Let  $q_1 = (q_1^1, q_1^2)$  and  $q_2 = (q_2^1, q_2^2)$  be the coordinates of the links and of the motors, respectively. Each link has length  $l_i$ , mass  $m_i$ , inertia  $I_i$  and distance to the center of mass  $r_i$ ; while each motor has associated inertia  $I_i$ ,  $i = 1, 2$ . We assume that the matrix  $I \in \mathbb{R}^{2 \times 2}$ , such that  $I = \text{diag}\{I_1, I_2\}$ , and the matrix  $K \in \mathbb{R}^{2 \times 2}$  is also diagonal of the form  $K = \text{diag}\{1/\varepsilon, 1/\varepsilon\}$ , thus fitting in the exposition above. Let us define the following constants

$$\begin{aligned} a_1 &= m_1 r_1^2 + m_2 l_1^2 + I_{l_1} \\ a_2 &= m_2 r_2^2 + I_{l_2} \\ b &= m_2 l_1 r_2. \end{aligned}$$

Then, the matrix  $M_l(q_1) \in \mathbb{R}^{2 \times 2}$  reads as

$$M_l(q_1) = \begin{bmatrix} a_1 + a_2 + 2b \cos q_1^2 & a_2 + b \cos q_1^2 \\ a_2 + b \cos q_1^2 & a_2 \end{bmatrix}.$$

The task is to make the links  $(q_1^1, q_1^2)$  follow a desired trajectory  $\bar{q}_{1,d} = q_{1,d} = (q_{1,d}^1, q_{1,d}^2)$  given by  $q_{1,d}^1 = q_{1,d}^2 = 0.1 + 0.05 \sin(t)$ . To achieve such a task, we implement separate controllers following the design principle of Kokotovic

et al. (1986). This is: one controller is designed for the rigid robot (the slow subsystem (19)) independently of the fast subsystem, and another controller is designed for the fast subsystem, where now the slow variables are taken as fixed parameters. These two controllers shall guarantee stability in their own domain (slow and fast reduced subsystems respectively). Then, the controller for the flexible-joint robot is defined as the sum of both reduced controllers. The stability of the flexible system is guaranteed by GSPT arguments, recall Section 2.3 and see Kokotovic (1984); Fenichel (1979); Jones (1995); Kaper (1999).

The controller synthesis for the rigid robot, which has a PH equation of the form (19) is taken from Dirksz and Scherpen (2013) and reads as

$$\begin{aligned} u_s &= M_l \ddot{q}_{1,d} + \frac{\partial}{\partial q_1} (M_l \dot{q}_{1,d}) \dot{q}_{1,d} - \frac{1}{2} \frac{\partial}{\partial q_1} (\dot{q}_{1,d}^T M_l \dot{q}_{1,d}) \\ &\quad - K_p(q_1 - q_{1,d}) - K_c(q_1 - q_{1,d} - q_{1,c}), \end{aligned} \quad (21)$$

where  $q_c$  is the controller state and its dynamics are given by

$$\dot{q}_{1,c} = K_d^{-1} K_c(q_1 - q_{1,d} - q_{1,c}),$$

and  $K_c$ ,  $K_d$  and  $K_p$  are positive definite matrices, see Dirksz and Scherpen (2013) for more details.

*Remark 6.* Eventhough (19) is not exactly the same model as considered in Dirksz and Scherpen (2013), both are, up to the change of coordinates of Section 2.2, a standard mechanical system. Moreover, note that in the change of coordinates (14)  $\bar{q}_\varepsilon = q_\varepsilon$ .

The controller  $u_s$  applied to the rigid robot has the performance shown in Figure 2a.

For the fast subsystem, which has the form (20), we employ the same controller design idea of Dirksz and Scherpen (2013). This is due to the fact that the layer subsystem is also a PH system, just now we have a desired trajectory  $\bar{q}_{2,d} = z_d = (z_d^1, z_d^2) = (0, 0)$ . The reason is that we want to follow the desired trajectory with a zero deflection, i.e.  $z = q_1 - q_2 = 0$ . This yields a controller of the form

$$u_f = -L_p z - L_c(z - z_c), \quad (22)$$

with the controller dynamics

$$\dot{z}_c = L_d^{-1} L_c(z - z_c),$$

where  $L_d$ ,  $L_p$  and  $L_c$  are positive definite matrices.

*Remark 7.* Note that the controllers  $u_s$  and  $u_f$  of (21) and (22), respectively, only use position measurements.

By combining these two controllers as  $u = u_s + u_f$ , and implementing them into the flexible-joint robot whose model is now of the form (17), we get the performance shown in Figures 2b and 2c. From these we see that the robot closely follows the desired trajectories after one second. Finally, comparing Figures 2a and 2b we note that the difference between the rigid and the flexible robot behaviors is barely noticeable.

#### 5. CONCLUSIONS

In this document we have explored the methodology of model order reduction based on singular perturbations for a PH system. In order to do so, we have written the PH model of a flexible-joint robot in a slow-fast format.



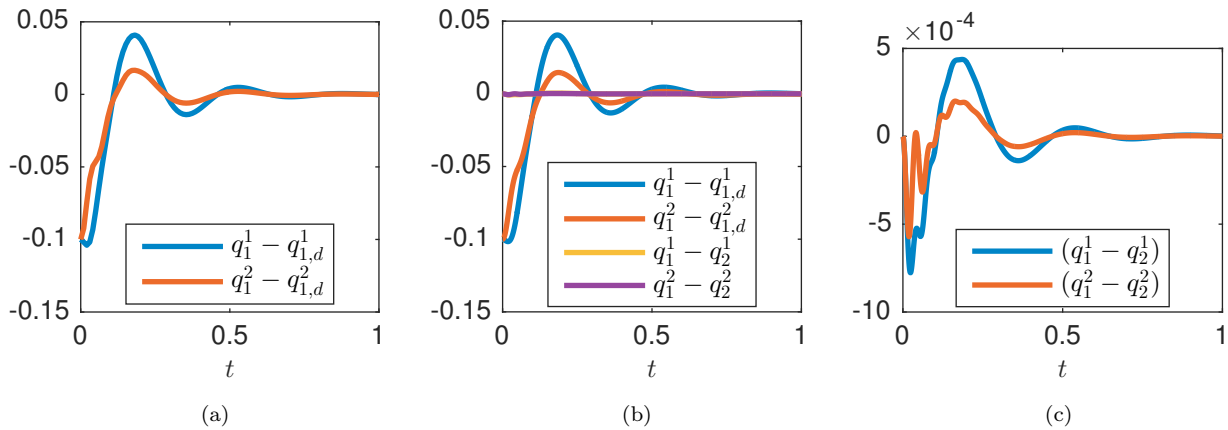


Fig. 2. (a) The trajectory tracking error for a *rigid* 2R robot. (b) The trajectory tracking error for a 2R *flexible-joint* robot. For this simulation  $\varepsilon = 0.01$  or  $K = \text{diag}\{100, 100\}$ . (c) Zoomed-in deflection.

Then, by inspecting the structure of the system we have shown, as it is to be expected, that the corresponding slow subsystem is the PH model of a rigid-joint robot. Consequently, as it happens in the EL framework, the design of controllers in the reduced subsystems is justified. To exemplify our exposition, we have implemented a controller with only position measurements designed in Dirks and Scherpen (2013). Our simulations show a good performance of the controllers. However, we have used controllers designed *only* on the reduced subsystems. The performance of the closed-loop systems can be improved by taking higher order terms of  $\varepsilon$  when designing the controller. In particular, and as a natural extension of this work, a careful study of the fast subsystem is required in order to rigorously prove exponential stability under composite controllers within the PH framework. Moreover, as future research, visco-elastic joints and elastic links may be incorporated to the PH model.

## REFERENCES

- Canudas-de Wit, C., Siciliano, B., and Bastin, G. (1996). *Theory of Robot Control*. Springer, London, UK.
- De Luca, A. (2014). Flexible Robots. In J. Baillieul and T. Samad (eds.), *Encyclopedia of Systems and Control*. Springer-Verlag London.
- Dirks, D.A. and Scherpen, J.M.A. (2013). On Tracking Control of Rigid-Joint Robots With Only Position Measurements. *IEEE Transactions on Control Systems Technology*, 21(4), 1510–1513.
- Duindam, V., Macchelli, A., Stramigioli, S., and Bruyninckx, H. (2009). *Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach*. Springer, Berlin, Germany.
- Fenichel, N. (1979). Geometric singular perturbation theory. *JDE*, 53–98.
- Fujimoto, K. and Sugie, T. (2001). Canonical transformation and stabilization of generalized Hamiltonian systems. *Systems and Control Letters*, 42(3), 217–227.
- Jones, C.K.R.T. (1995). Geometric singular perturbation theory. In *Dynamical Systems*, LNM 1609, 44–120. Springer-Verlag.
- Kaper, T.J. (1999). An Introduction to Geometric Methods and Dynamical Systems Theory for Singular Perturbation Problems. In *Symposia in Applied Mathematics*, volume 56, 85–131. AMS.
- Kokotovic, P.V. (1984). Applications of Singular Perturbation Techniques to Control Problems. *SIAM Review*, 26(4), 501–550.
- Kokotovic, P.V., O'Reilly, J., and Khalil, H.K. (1986). *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, Inc., Orlando, FL, USA.
- Kokotovic, P., O'Malley, R., and Sannuti, P. (1976). Singular perturbations and order reduction in control theory An overview. *Automatica*, 12(2), 123–132.
- Murray, R.M., Zexiang, L., and Sastry, S.S. (1994). *Mathematical Introduction to Robot Manipulation*. CRC, USA.
- Ortega, R., Loria, A., Nicklasson, P.J., and Sira-Ramirez, H. (1998). *Passivity-Based Control of Euler-Lagrange Systems*. Springer, London, UK.
- Polyuga, R.V. and van der Schaft, A. (2010). Structure preserving model reduction of port-hamiltonian systems by moment matching at infinity. *Automatica*, 46(4), 665 – 672.
- Scherpen, J. and van der Schaft, A. (2008). A structure preserving minimal representation of a nonlinear port-hamiltonian system. In *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, 4885–4890.
- Spong, M., Hutchinson, H., and Vidyasagar, M. (2006). *Robot modeling and control*. Wiley, USA.
- Spong, M.W. (1987). Modeling and Control of Elastic Joint Robots. *Journal of Dynamic Systems, Measurement, and Control*, 109(4), 310.
- van der Schaft, A.J. (2000). *L<sub>2</sub>-Gain and Passivity Techniques in Nonlinear Control*. Springer, London, UK.
- van der Schaft, A.J. and Maschke, B.M. (2003). Port-Hamiltonian systems: a theory for modeling, simulation and control of complex physical systems. *Unibo*.
- van der Schaft, A. and Polyuga, R. (2009). Structure-preserving model reduction of complex physical systems. In *Proceedings of the 48th IEEE Conference on Decision and Control, 2009*, 4322–4327.
- Verhulst, F. (2005). *Methods and Applications of Singular Perturbations*, volume 50 of *Texts in Applied Mathematics*. Springer.
- Viola, G., Ortega, R., Banavar, R., Acosta, J.A., and Astolfi, A. (2007). Total energy shaping control of mechanical systems simplifying the matching equations via coordinate changes. *IEEE Transactions on Automatic Control*, 52(6), 1093–1099.